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Parameter estimation of a nonlinear benchmark system

Juš Kocijan*

Jozef Stefan Institute, Ljubljana, Slovenia and *University of Nova Gorica, Nova Gorica, Slovenia*

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ABSTRACT

This paper describes a possible procedure for the parameter estimation of a nonlinear benchmark problem called Silver box under the presumption that it has a structure of Duffing oscillator. The multiple-minima problem of optimisation is tackled with a combination of stochastic and deterministic optimisation methods. Simulation responses of the obtained model show a good match with the measurements taken from the electronic device, which is confirmed by validation of input/output response and residuals analysis.

Keywords: *mathematical modelling, parameter estimation, nonlinear benchmark.*

INTRODUCTION

Nonlinear system modelling is a very important field in system theory and also increasingly in control engineering practice. There is no successful control design, fault diagnosis, or dynamic systems analysis without a good model of the considered system. The modelling of dynamic systems is, in general, divided on first-principles modelling, which is modelling based on physical laws behind the system functioning, on experimental modelling or system identification, which is modelling from measured data, and on a combination of both approaches. Commonly, the combination of first-principles modelling and system identification is used, in which the model structure is obtained from physical laws and then missing model parameters are estimated from measurements.

The development of new modelling methods, especially system identification methods, is a challenging task. It is necessary to evaluate and compare newly developed algorithms on measurement data in a way that is repeatable not only to the developer but also to users of the modelling algorithm. Such reference data that enable comparison are called benchmarks. A number of properties of the benchmark data set are desirable.

**Correspondence to: Jus Kocijan, Jozef Stefan Institute, Jamova cesta 39, SI-1000, Ljubljana, Slovenia, E-mail: jus.kocijan@ijs.si*

These properties should enable the comparison of models obtained during algorithm development to a reasonably well-known true result. One of the selections of benchmark data for nonlinear system identification is [1]. This web page contains eight data sets representing behaviour of nonlinear dynamic systems.

While [1] contains only data sets and a description of the systems from which the measurements were obtained, it is useful to get more insight into the dynamics behind the systems. In this paper, our intention is focused on finding a model for one of the benchmarks with presumed structure and unknown parameters. The case study and benchmark considered in the paper concerns an electronic implementation of a nonlinear system, denoted as 'the Silver box' [2, 3].

The analogue electrical circuit represents a nonlinear mechanical resonat ing system with a moving mass m, a viscous damping d, and a nonlinear spring k(y) as shown in Fig. 1. The purpose of the electrical circuit is to relate the displacement y(t) to the force u(t) according to the following differential equation:

$$m\frac{d^2y(t)}{dt} + d\frac{dy(t)}{dt} + k(y(t))y(t) = u(t) \quad (1)$$

The nonlinear spring is described by using a static position-dependent stiffness

$$k(y(t)) = a + by^2(t) \quad (2)$$

Such a nonlinear system is often used in the literature to demonstrate properties of different modelling methods [4], [5], [6], [2], [7], etc., even though not everybody uses the very same system.

Equation (1) describes what is known as the Duffing oscillator. It is a periodically forced oscillator with a nonlinear elasticity. It is also known as an example of a simple chaotic model. The Duffing oscillator can be interpreted as a forced oscillator for b > 0. When the spring constant is a > 0, it is called a hardening spring. More details can be found in [6] or [8]. The data used in this

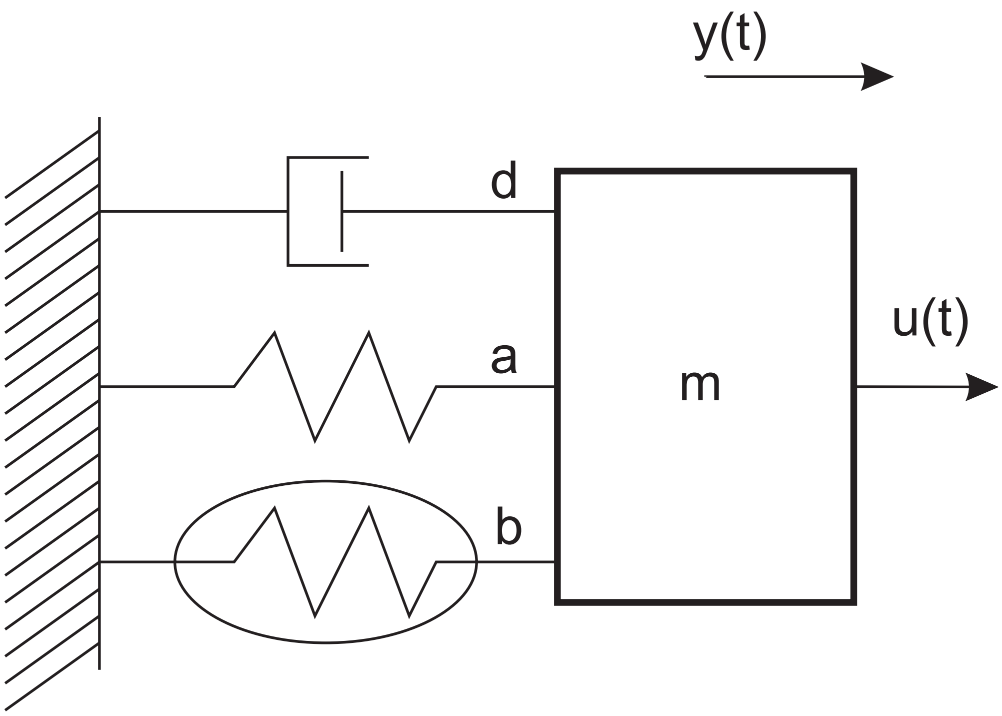

Fig. 1. Scheme of the nonlinear spring represented by the Silver box benchmark.

study are available online [3]. The sampling time of the voltage signals' measurements is $T_s = 2^{14}/10^7 = 0.0016$ s [9]. The input signal to electronic circuit u(k), where k are time instants, consists of two parts.The first part (40 000 samples) is a white Gaussian noise sequence filtered by a Butterworth filter. Its amplitude is changed linearly over the interval from zero to its maximum value. The second part of the signal consists of 10 successive realisations of a random multi-sine signal. The amplitude of the input signal is scaled such that the signal fits well within the range of the generator. Ten realisations of the random multisine were merged, each of them separated by 100 zeros to indicate the start of a new realisation. More details can be found in [9]. The data generated by the electronic system is shown in Fig. 2.

As the benchmark problem from [3] comes without the precise description of the system from which measurements were collected, the values of the parameters in Equation (1) are not known. The purpose of this paper is to demonstrate estimation of the parameters of the Silver-box benchmark system presuming that it corresponds to the structure of Equation (1). For the purpose of data estimation, 10 000 data points were used for training and 4000 for testing. The parameter estimation was pursued with Matlab software.

The paper is structured as follows. The modelling method, or more precisely the parameter-estimation method, is described in the next section. Section 3 deals with the obtained results. The conclusions are gathered at the end of the paper.

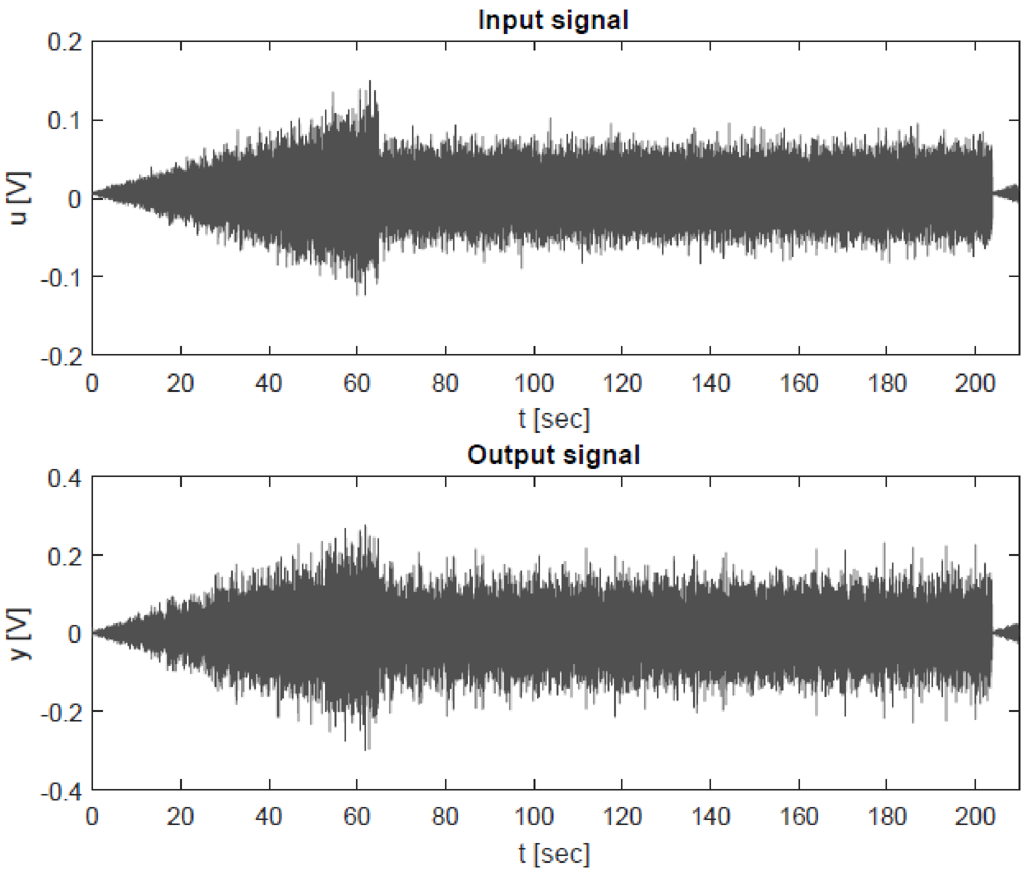

Fig. 2. Available input and output benchmark signals.

Modelling method

The parameters of a dynamic system can be estimated in various ways and it depends on the modeler's objectives and the system at hand. In our case,the selected cost function is a square of errors because it exaggerates and differentiates between large and small errors, because it has other nice mathematical properties, and because it is frequently used. Other cost functions can be used as well.

In our case, we estimate only four parameters, but the manifold of cost function has multiple, local minima, which means multiple, locally optimal points of the optimisation-problem solution in which the optimisation can get stuck. The optimisation-problem solution is very sensitive to the selection of initial values for optimised parameters, and was tested with several optimisation attempts with the local optimisation method. Consequently, the stochastic and global optimisation method, like the Genetic algorithm, Differential evolution, Particle swarm optimisation, etc., seems to be a convenient tool to find optimal parameters.

A common characteristic among all stochastic optimisation methods is that, although they come close, they do not provide the optimal solution in a single run. There are two ways to deal with this issue. One is that one does multiple runs of the selected method and looks for the mean solution. The other is to use the solution of the stochastic optimisation method as the initial values for one of the deterministic and local optimisation methods that bring the solution to the optimal value. In our case, the later approach was selected.

The *Differential evolution method* was, in our case, selected as a stochastic optimisation method due to personal familiarity with the method and its good properties. The minimum search with the *gradient descent method*, in particular the interior-point algorithm, was used as the deterministic optimisation method.

Differential evolution is a method for numerical optimisation without explicit knowledge of the gradients. It works on multidimensional, real-valued functions that are not necessarily continuous or differentiable, and was introduced by Storn and Price [10]. The method searches for a solution to a problem by maintaining a population of candidate solutions and creating new candidate solutions by combining existing ones according to its simple for mulae of vector

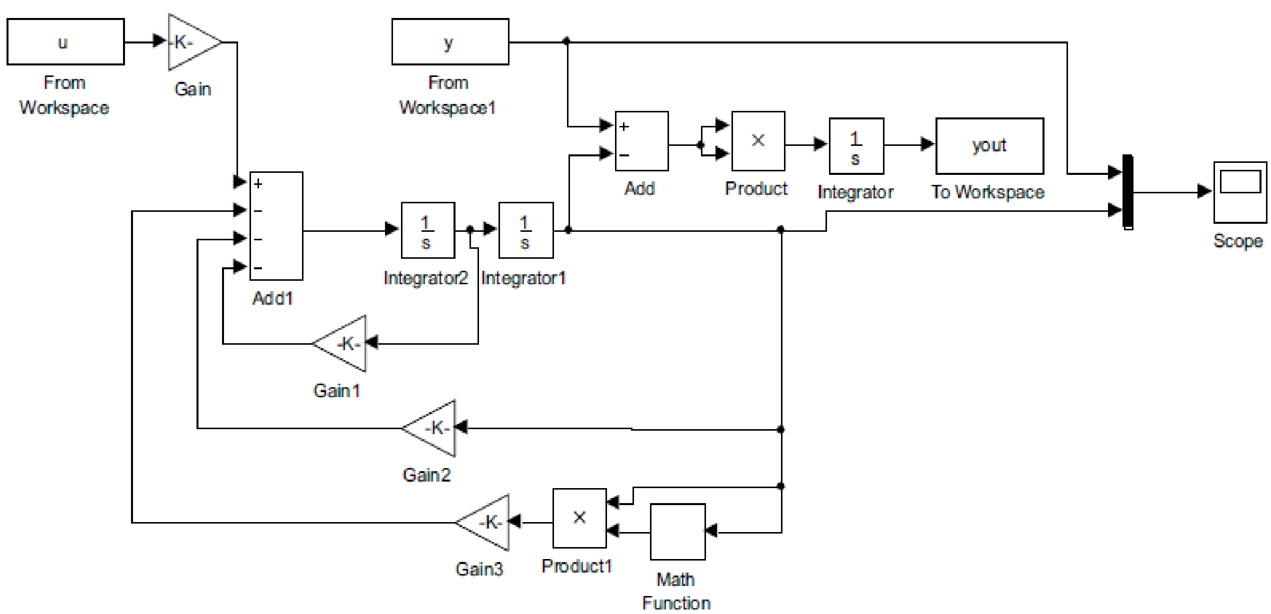

Fig. 3. Simulation scheme in Simulink used for parameter optimisation.

crossover and mutation. It then keeps whichever candidate solution has the best score or fitness on the optimisation problem at hand. The optimisation problem is treated as a black box that merely provides a measure of quality given a candidate solution and, therefore, the gradient is not needed. More details about Differential evolution can be found in [11].

The **gradient descent method** is a well known family of deterministic, and at the same time local optimisation, methods. And, the interior-point algorithm is the one that is already implemented within Matlab software. Optimisations were pursued with Matlab with embedded Simulink simulation. The simulation scheme that incorporates model structure as well as cost function is depicted in Fig. 3. The model was simulated with a variable integration step in each run. Parameters for stochastic optimisation was as follows: the population size was set to 50 individuals and the number of iterations to 1000.

RESULTS AND DISCUSSION

The obtained estimated-parameter results of stochastic optimisation that were used in the next stage as initial values for deterministic optimisation are as follows:

$1/m$ = 197970,
d/m = 43,
a/m = 193507,
b/m = 789822. (3)

The parameters correspond to the electronically implemented system and therefore cannot be characterised with measurement units. It is not clear whether the unit for mass, for example, is expressed with equivalents to kilograms, tones, or perhaps some non SI system units. Consequently, the parameters are depicted without measurement units.

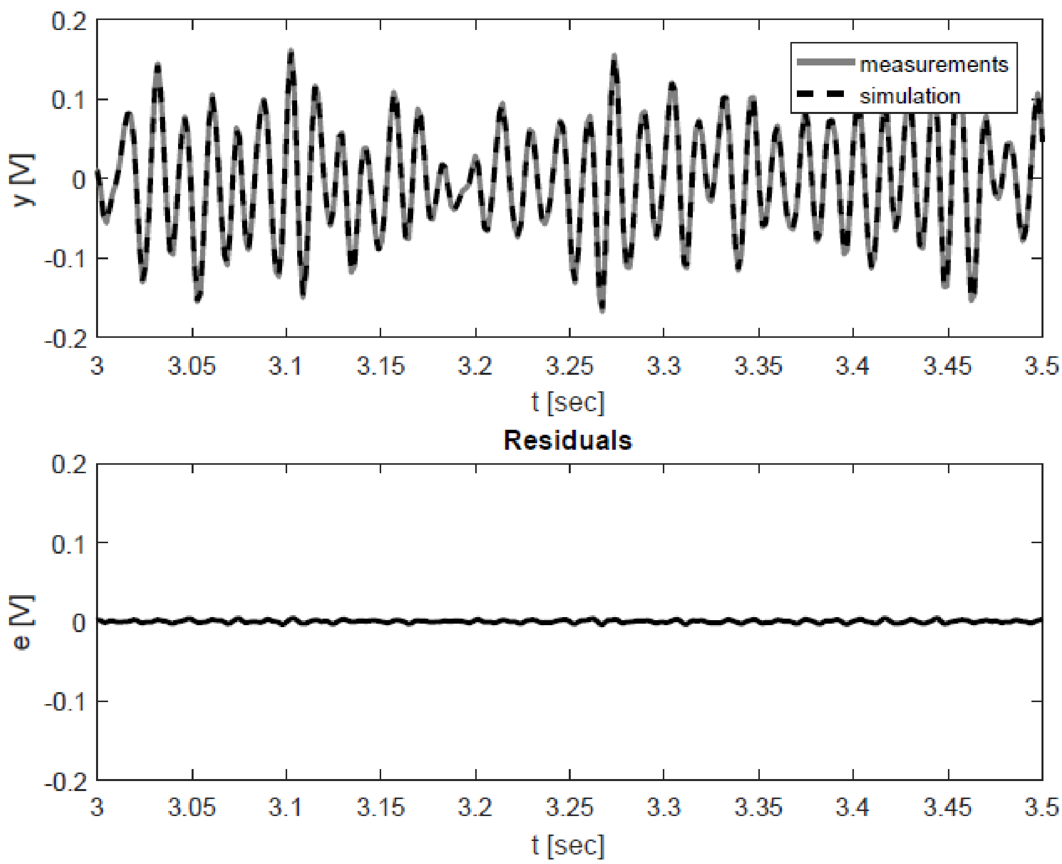

Fig. 4. Model simulation response versus validation signal and corresponding residuals.

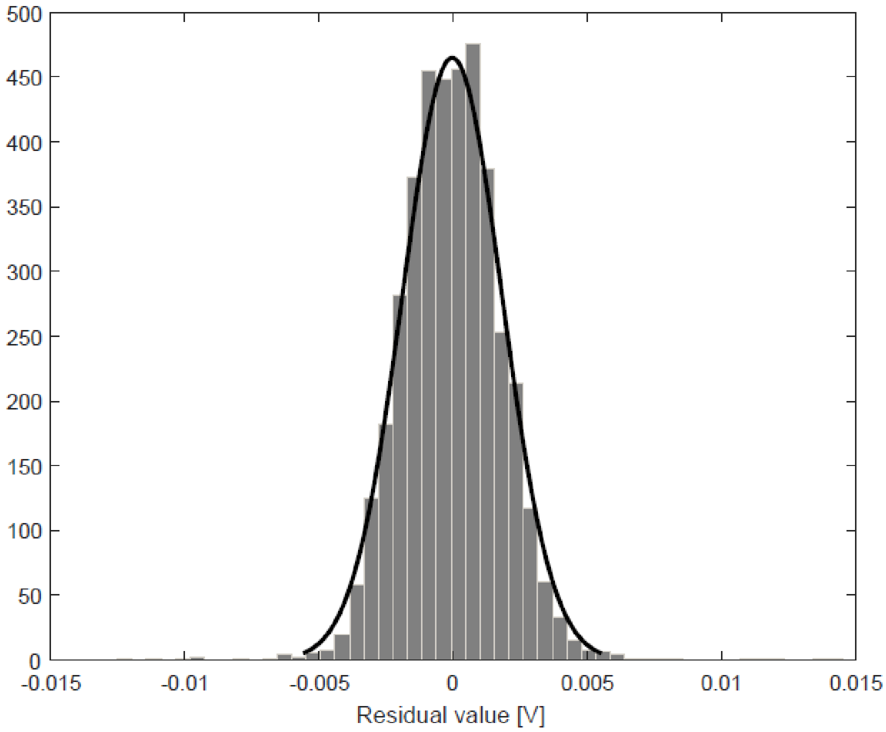

Fig. 5. Distribution of residuals for the validation signal.

The final values of the deterministic optimisation are as follows:

$1/m = 199875$,
$d/m = 43$,
$a/m = 193471$,
$b/m = 794733$. (4)

And, so, the estimated parameters of the Equation (1) are as follows:

$m = 5 \cdot 10^{-6}$,
$d = 2.151 \cdot 10^{-4}$,
$a = 0.968$,
$b = 3.976$. (5)

The model was validated with a simulation on the data that was not used for parameter estimation. Fig. 4 shows a detail of response for a 5 second interval. It is clear that the simulation response of the model with estimated parameters fits the measured data on the validation signal very well.

To validate model simulation response for the entire validation signal,the normalised root-mean-square-error (NRMSE) criterion is used:

$$\mathrm{NMRSE} = 1 - \frac{\|\mathbf{y} - \hat{\mathbf{y}}\|^2}{\|\mathbf{y} - E(\mathbf{y})\|^2}, \quad (6)$$

where y is the vector of validation values, ŷ is the vector of model response values, and E(y) is the mean value of y. NRMSE has value 1 for a perfect match and $-\infty$ for an extremely bad match of validation and mean predicted values. The obtained result for the entire validation signal is NMRSE = 0,971 = 97,1 %;which can be considered as a very good fit.

The residuals left from the estimated model shall be white noise representing measurement noise. The distribution of residuals fitted with a Gaussian function is shown in Fig. 5. It is clear from Fig. 5 that the residuals are white noise. The phase plot of the model with estimated parameters is shown in Fig. 6.

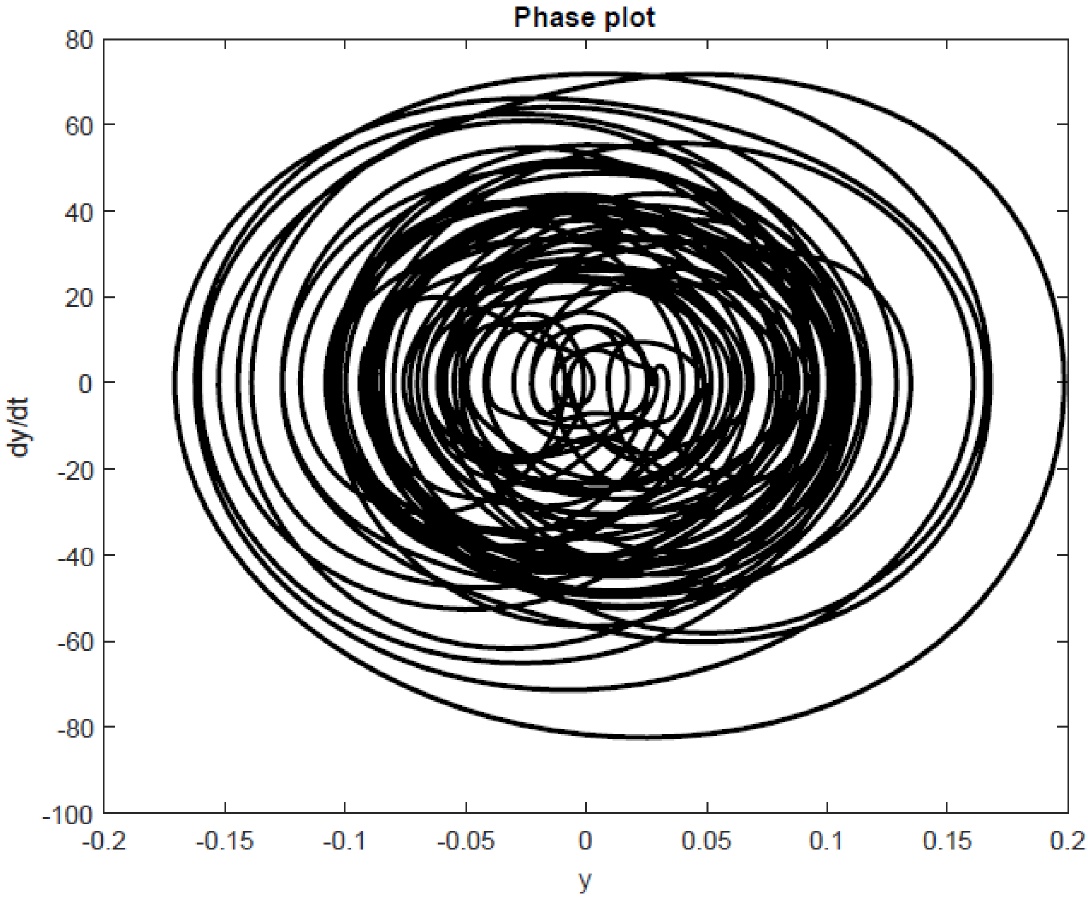

Fig. 6. Phase plot of the model with estimated parameters.

CONCLUSIONS

A possible procedure for the parameter estimation of a nonlinear benchmark problem was presented in the paper and illustrated with available data. The multiple-minima problem of optimisation was tackled with the combination of a stochastic and a deterministic optimisation method.

Simulation responses of the obtained model show a good match with the measurements taken from the electronic device named Silver box, which is confirmed by validation of input/output response and residuals analysis.

The obtained model may be used for various analysis of modelling and control methods with the Silver box benchmark, which have been, until present, limited to methods that can exploit available measurements.

Acknowledgements

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